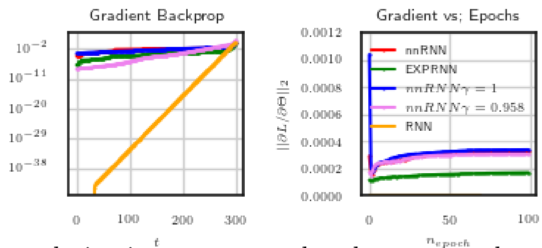
We highlight the delightful situation where reviews considerably enhanced the impact of our results without any structural changes to the story, and thank the reviewers for it. Importantly, R3's remarks led to an expanded interpretation of non-normality in RNNs which, combined to our proposed nnRNN model, we believe are of important significance to the understanding of RNN gradients. An updated version of our paper incorporates the points below.

R1: Q"*Add more explanation and insight on why the gap to LSTM performance is still larger than that to expRNN.*" **A:** We believe this is due to the task design which suits gated networks very well (LSTM performs poorly on the other two tasks tested). As indicated in the main text, our goal is to perfect parametrization of recurrent connectivity, which can then be combined to gated architecture (ongoing and future work).

R2: Q"*[Clarify] how the asymptotic (in number of units or parameters) runtimes of the different approaches compare. Especially the runtime of optimizing P.*" **A:** The forward pass of the nnRNN has the same complexity as that of a vanilla RNN, that is $O(Tn^2)$, for a hidden state of size n and a sequence of length T. The backward pass is similarly $O(Tn^2)$ plus the update cost of P, in addition to a once-per-update cost of $O(n^3)$ to combine the Schur parametrization via matrix multiplication. Importantly, the nnRNN leverages any orthogonal/unitary optimizer fof P which have complexities ranging from $O(n \log n)$ to $O(n^3)$ at each update, with their own advantages and caveats (see related work section in main text). We chose the expRNN scheme which is $O(n^3)$ in the worst case, but has fast run-time in practice.

R3: Q"*There is no guarantee that nnRNN solves the exploding gradient problem. [...] comparison [of] nnRNN with spectral RNN might provide important insights. I recommend adding the theoretical or empirical analysis of the exploding gradient problems.*" **A:** We thank R3 for pointing us to the *spectral RNN* (Zhang, Lei, and Dhillon, ICML 2018), which presents an SVD decomposition with the same motivation as our to Schur-decomposition: to maintain expressivity, whilst controlling a sprectrum (both using regularization). R3 astutely remarks that "*[the relationship between these methods could] reveal whether we should constrain the eigenvalues or singular values of recurrent connection matrices*". We strongly believe that this distinction is important and not well understood by the community, and that the changes described herein add clarity to this question, in addition to strengthening the existing message of our paper.

As pointed out, constraining the eigenspectrum to the unit circle mitigates gradient vanishing, but not necessarily gradient explosion, as singular values can still be greater than one (and so too the spectral norm of Jacobians). In this case however, gradients explode polynomially in time rather than exponentially (Pascanu et al. (2013), Arjovsky et al. (2015)). We provide a theorem to establish this for triangular matrices. **Theorem:** Let $A \in R^{n \times n}$ be a matrix such that $A_{ii} = 1$, $A_{ij} = x$ for $i < j$, and $A_{ij} = 0$ otherwise. Then for all $d \geq 1$ and $j > i$, we have $(A^d)_{ij} = p^{(d)}_{j-i}(x)$ is polynomial in x of degree at most $j - i$, where the coefficient of x^0 is zero and the coefficient of x^l is $O(\binom{d}{l})$ for $l = 1, 2, \ldots, j - i$. (proof by induction will be presented in appendix) This reveals that: **(1) Gradient explosion in nnRNN, if present, is not as severe as if eigenvalues were larger than one**. As shown below, training of nnRNN with eigenvalues strictly on the unit circle may be successful, albeit somewhat unstable, but still more expressive than orthogonal RNNs. The figure shows that gradients for nnRNN on PTB task, with eigenvalues clamped or regularized, behave nicely during backpropagation and throughout training.



(2) For a matrix with eigenvalues on the unit circle, non-normality necessarily implies a largest singular value greater than one. Thus, the expressivity afforded by non-normality must come with a trade-off between maintaining the eigen and singular spectra "close" to the unit circle, balancing control over exponential vanishing and polynomial exploding gradients respectively. This fact remains true for any parametrization of non-normal matrices, including the SVD used in spectral RNN. The nnRNN is naturally suited to target this balance by explicitly allowing regularization over normal and non-normal parts of a matrix, and enabling the optimizer to find that trade-off. This explains why, in the main text, we find that allowing eigenvalues to deviate slightly from the unit circle throughout training (regularization on γ), along with weight decay for the non-normal part, yields the best results with most stable training. Further evidence of this balancing mechanism is found in trained matrices (see Fig 3 in main text). For the PTB task, non-normal structure emerges and the mean eigenvalue norm is balanced at $\bar{\gamma} \sim 0.958$. In contrast for the copy task, matrices remain normal and $\bar{\gamma} \sim 1$. Our Schur approach complements that of the SVD approach in additional ways: by adding the freedom to distribute eigenvalues on the unit circle (which we showed was exploited by the learning), and by adding interpretability as interaction between modes (which revealed task-intuitive differences in the solutions).

R3:"*Since optimization of the gamma obscures the cause of improvements, you should compare nnRNN with gamma=1 to other methods.*" **A:** Thanks to the results above, we now know that optimization of γ plays an intricate role in allowing the expressivity afforded by non-normal connectivity structure while conserving good gradient propagation. Nevertheless, we acknowledge that it confounds the expressive role of non-normality. To elucidate this, we train the nnRNN on PTB by clamping γ at 1, and at 0.958 (the mean values found by the optimizer in the unclamped case) respectively. Results complement those of Table 1 in the main text ($\sim$1.32M params, $N = 1024$ units). As expected for $\gamma = 1$, some run did not converge (asterisks indicate number out of 5) as the emergence of non-normal structure pushes singular values above one. Despite this, on runs that did converge we found the best performance out of all methods (including regularized γ nnRNN), strongly indicating that non-normality does indeed provide more expressivity. For γ clamped at 0.958 the performance was virtually identical to that of nnRNN with regularized γ, indicating non-normal connectivity learning appears robust and independent of γ learning. Exploration of novel ways to promote the balance between polynomial explosion and exponential vanishing is promising future work.

Model	$T_{PTB} = 150$	$T_{PTB} = 300$
nnRNN-$\gamma = 1$	1.46 $\pm$ 0.005*	1.49 $\pm$ 0.022**
nnRNN-$\gamma = 0.958$	1.47 $\pm$ 0.005	1.49 $\pm$ 0.008